D. Getting to the Wave Equation

(a) Wave Equation? What Wave Equation? What does a Wave Equation do?

Take a familiar example: EM wave equation in vacuum

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{(same form for } \vec{B}\text{)} \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

For a standard form†

$$\vec{E} = \vec{E}_0\, e^{i\left(\frac{2\pi x}{\lambda} - 2\pi f t\right)} = \vec{E}_0\, e^{i(kx - \omega t)}$$

- Substituting into wave equation gives the dispersion relation

$$k^2 = \frac{\omega^2}{c^2} \quad \text{OR} \quad \omega = ck \quad \text{OR} \quad f\lambda = c$$

† In EM, the complex form is for convenience. We could use $\sim \cos(kx - \omega t)$.

- At $t=0$, $\vec{E}(x,t=0) = \vec{E}_{0,1}\, e^{ik_1 x} + \vec{E}_{0,2}\, e^{ik_2 x}$

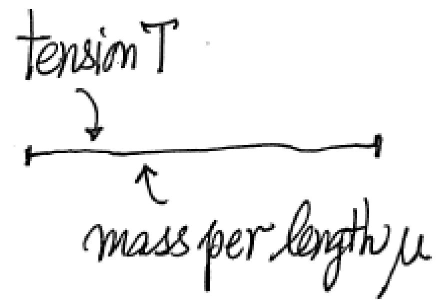
then $$\vec{E}(x,t) = \vec{E}_{0,1}\, e^{ik_1 x} \underbrace{e^{-i\omega_1 t}}_{\omega_1 = ck_1} + \vec{E}_{0,2}\, e^{ik_2 x} \underbrace{e^{-i\omega_2 t}}_{\omega_2 = ck_2} \qquad (*)$$

Ex: Show that $(*)^{\dagger}$ is a solution to the EM wave equation

Remark: This comes about because the wave equation is a <u>Linear Partial Differential Equation</u>

- Another interesting example:

Guitar String $$\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{\mu}{T}\right) \frac{\partial^2 \psi}{\partial t^2}$$



Boundary Conditions select normal modes

(b) Quantum Particle (massive)

de Broglie: $\lambda_{dB} = \dfrac{h}{p} = \dfrac{2\pi\hbar}{p}$

- Particle with a definite p $\Rightarrow$ there is a (one value) definite λ
- <u>Free particle</u>
  - No force, constant potential energy over space $-\dfrac{\partial U}{\partial x} = F = 0$
  - Take $U = \text{constant} = 0$
  - $E = \frac{1}{2}mv^2 = \dfrac{p^2}{2m}$ $\quad\therefore\quad \boxed{E = \dfrac{p^2}{2m}}$

plays the role of the dispersion relation

Recall: Using photon properties, $\vec{E} \sim \vec{E}_0 e^{ikx} e^{-i\omega t}$

$$\sim \vec{E}_0 e^{i\left(\frac{px}{\hbar} - \frac{Et}{\hbar}\right)}$$

- Massive (mass m) particle of definite momentum p (free particle)

Wave form $\Psi_p(x,t) \sim e^{i\left(\frac{2\pi x}{\lambda} - \omega t\right)} \sim e^{i\left(\frac{px}{\hbar} - \frac{Et}{\hbar}\right)}$

emphasizes definite p

and E and p are related by $E = \frac{p^2}{2m}$

The complex wavefunction for a free particle in QM is ***a necessity*** because a definite momentum implies Δp=0. The complex form gives |ψ|^2 = constant everywhere, thus corresponding to Δx -> ∞, as required. A sinusoidal wave will not work.

$\boxed{\text{What is a Wave Equation that would give the right } E\text{-}p \text{ relation?}}$